MATH 155 - Chapter 9.3 - The Integral Test and p-Series:
(Can only be applied to positive series)

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1. **Theorem: The Integral Test**

   Let f be a continuous, positive, decreasing function on the interval $[1, \infty)$ and suppose that $a_n = f(n)$ for all positive integer n.
   Then the infinite series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ converges.
   Also, $\sum_{n=1}^{\infty} a_n$ diverges if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ diverges.

   Note: The integer n may be replaced by any positive integer throughout this theorem.

2. **Important Note:**
   The theorem above only guarantees convergence of $\sum_{n=1}^{\infty} a_n$ if $\int_1^{\infty} f(x)\,dx$ converges. It does NOT mean that $\sum_{n=1}^{\infty} a_n = \int_1^{\infty} f(x)\,dx$.

3. **Definition: (p-Series)** The series of the form

   $$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \qquad \text{is called the } p\text{-series.}$$

4. **Definition: (Harmonic Series)** The series of the form

   $$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

   is called the **harmonic series. The harmonic series is divergent.**

5. **Theorem: The Convergence of p-Series**

   The p-series

   $$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

   1. Converges if $p > 1$.
   2. Diverges if $0 < p \leq 1$.